

An application of Grossone to the study of a family of tilings of the hyperbolic plane

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Abstract

In this paper, we look at the improvement of our knowledge on a family of tilings of the hyperbolic plane which is brought in by the use of Sergeyev's numeral system based on grossone, see [17, 18, 19]. It appears that the information we can get by using this new numeral system depends on the way we look at the tilings. The ways are significantly different but they confirm some results which were obtained in the traditional but constructive frame and allow us to obtain an additional precision with respect to this information.

1 Introduction

This paper gives an application of the new methodology introduced by Yaroslav SERGEYEV in his seminal papers, see [17, 18, 19], to the study of a family of tilings of the hyperbolic plane.

The hyperbolic plane is of great interest by itself: both for theoretical reasons, as Section 3 will convince the reader and also for practical ones. Up to now, the main applications of hyperbolic geometry are theoretical and they concern the theory of relativity. Recently, a few applications were planned and a few of them realized, we refer the interested reader to [9, 11]. From what is explained in [9, 11], we can infer that hyperbolic geometry, and especially the location technique of tiles described in [8] and in Section 4, can be of use for issues involving huge nets.

In Section 2, we remind the basic features of the new numeral system which allows to deal with infinite sets. Then, in Section 3 we remind what is needed of hyperbolic geometry in order to introduce the family of tilings which we consider in Section 4. In Section 5, we state the results and prove them. The results

presented in this paper are slightly different from those briefly presented at the International Workshop *Infinite and Infinitesimal in Mathematics, Computing and Natural Sciences* held at Cetraro, Italy, in May 2010, see [13]. They are in some sense a more precise version of what was given in [13]. In Section 6 we indicate a few possible continuations.

2 The new numeral system

In papers [17, 18, 19], Yaroslav SERGEYEV gives the main arguments in favour of the new numeral system he founded, allowing to obtain more precise results on infinite sets that what was obtained previously.

We can sum up the properties of the system as follows.

We distinguish the **objects** of our study from the **tools** we use to **observe** them. These three parts of the knowledge process have to be more clearly distinguished as they were traditionally in mathematics, contrarily to other domains of science, as physics and natural sciences where this distinction is clearly observed. This is the content of *Postulate 2* in the quoted papers. It is an important issue for mathematics where the distinction between an **observer** and what is **observed** is very often forgotten. In particular, not enough attention is paid to subjectivity of the observer and the relative validity of his/her observations. The latter are very dependent of cultural elements, especially the *language* used by the observer to describe what he/she sees.

We are interested in the properties of the objects, some of them being possibly infinite or infinitesimal, but **operations** on the objects, performed by a human being or a machine, necessarily deal with **finitely many** of them and only **finitely many** operations can be applied within the frame of an argument. This is the content of *Postulate 1* in the quoted papers.

At last and not the least, we consider that the principle *The part is less than the whole* has to be applied to all numbers, finite, infinite or infinitesimal, and also to all sets and processes, whether finite or infinite. This is the content of *Postulate 3* of [17, 18, 19].

On the basis of these principles, Yaroslav SERGEYEV introduced a new numeral system in order to be able to write down infinite numbers. To this aim, an **infinite natural number** is introduced, **grossone**, denote by $\mathbb{1}$, which is the number of elements of the set of positive integers. This number satisfies the following three properties which are axioms of the system:

- for any finite natural number n , $n < \mathbb{1}$.
- we have $0 \cdot \mathbb{1} = \mathbb{1} \cdot 0 = 0$, $\mathbb{1} - \mathbb{1} = 0$, $\frac{\mathbb{1}}{\mathbb{1}} = 1$, $\mathbb{1}^0 = 1$, $1^{\mathbb{1}} = 1$ and $0^{\mathbb{1}} = 0$.
- let $N_{k,n}$ be the set of positive integers of the form $k+jn$, with k and n positive finite integers, $k < n$, for j running over the set of the positive integers; notice that these sets are pairwise disjoint and that their union is the set of all positive integers; then all these sets have the

same number of elements denoted by $\frac{\textcircled{1}}{n}$.

Denote by \mathbb{N} the set of positive natural numbers. All traditional operations performed on natural numbers are extended to $\textcircled{1}$ in a natural way with the standard properties, among them: commutativity and associativity of addition and multiplication and distributivity of multiplication over addition. As n is the number of elements of the set of the finite positive integers from 1 to n , and as $\textcircled{1}$ is, by definition, the number of elements of \mathbb{N} , a consequence of the properties of addition and multiplication, is that \mathbb{N} also contains a lot of other infinite numbers: all of them of the form $\frac{\textcircled{1}}{n}$ and, more generally, all the numbers $\frac{j\textcircled{1}}{n} \pm k$ for any $j \in 1..n$ and any finite natural number k , n being any positive integer. From now on, we shall call **infinite numeral system** the system described as above.

Before turning to hyperbolic geometry and our application of this system to them, we conclude this short introduction to the infinite numeral system by two points about infinite numbers.

First, let us remark that there are other infinite numbers as those described in the previous paragraphs. Let us remark that we can define numbers by defining their properties. We know that any finite positive number n is the greatest element of the set of positive numbers m such that $m \leq n$. This definition can in fact be extended to $\textcircled{1}$ itself which is the number of positive numbers, so that $\textcircled{1}$ itself is a number and from what is just said, it is the greatest of them. As an other example, consider the set S of positive integers x such that $x^2 \leq \textcircled{1}$. The number κ of elements of S is also the greatest element of S , by analogy with what we have seen with any finite n and with $\textcircled{1}$ itself. Now, κ is infinite. Otherwise, κ being finite would entail that $\kappa+1$ would also be finite and so, $(\kappa+1)^2$ would also be finite. Accordingly, $\kappa+1$ would belong to S , a contradiction with the maximality of κ . From this, we obtain that $\textcircled{1} < (\kappa+1)^2$, so that we can write that $\kappa = \lfloor \sqrt{\textcircled{1}} \rfloor$. We shall go back to this way of defining numbers in Section 5.

Second, we shall also use **sequences** of numbers. A **sequence** of elements of a set A is a mapping from the set of positive integers into a set A . As a consequence of the above axioms, the number of elements of a sequence is at most $\textcircled{1}$. We say that a sequence is **complete** if it exactly has $\textcircled{1}$ elements.

3 Hyperbolic geometry

Hyperbolic geometry appeared in the first half of the 19th century, proving the independence of the parallel axiom of Euclidean geometry. Models were devised in the second half of the 19th century and we shall use here one of the most popular ones, Poincaré's disc. This model is represented by Figure 1.

In the figure, the model works as follows: inside the open disc, we have the points of the hyperbolic plane. Lines are trace of diameters or circles orthogonal to the border of the disc. As an example, the line m of Figure 1 is such a line.

Through the point A we can see a line s which cuts m , two lines which are parallel to m : p and q , touching m in the model at P and Q respectively. The points P and Q are points of the border. They do not belong to the hyperbolic plane and, for this reason, they are called points at infinity. At last, and not the least: the line n also passes through A without cutting m , neither inside the disc nor outside it. This line is called **non-secant** with m . Two lines are non-secant if and only if they have a common perpendicular which is unique.

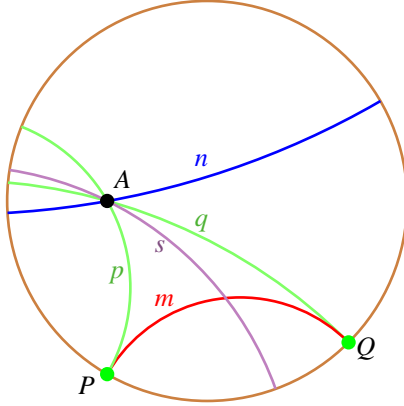


Figure 1 *Illustration of the parallel axiom of hyperbolic geometry in Poincaré's disc model.*

The model can be generalized to any dimension, but as we deal with the plane only in this paper, we simply refer the reader to [8, 9] where dimensions 3 and 4 are studied for further indications. More classical approaches can be found in [14], [15] and [16], for instance. Hyperbolic geometry is used in the theory of relativity, see [23, 22] and in several cosmological models, see [2]. We refer the interested reader to the corresponding sections of *Wikipedia* for more references. In the first subsection of the next section, we indicate more specific applications of tilings of the hyperbolic plane.

4 The tilings $\{p, q\}$

We remind the reader that a **tiling** is a partition of a geometrical space X where the closures of the elements of the partition are supposed to be obtained from a set \mathcal{S} of parts of X by isometries of the space. We say that \mathcal{S} is the set of **prototiles**. The closures of the elements of the partition are said **copies** of the prototiles and they are called **tiles**. Moreover, there can be an additional condition on the abutting tiles to be satisfied: they are called the **matching** conditions.

In this paper, we shall focus on the case where we have finitely many prototiles which are all copies of the same polygon P . Such a tiling is called a

tessellation when it is generated by reflection in the sides of P and, recursively, of the images in their sides. Below, Figure 2 illustrates two particular cases of tessellations to which we turn a bit later.

4.1 Poincaré's theorem

First, we mention an important theorem proved by Poincaré which says that there are infinitely many different polygons giving rise to a tessellation of the hyperbolic plane.

Theorem 1 (Poincaré) – *A triangle of the hyperbolic plane whose angles are of the form $\frac{2\pi}{p}, \frac{2\pi}{q}$ and $\frac{2\pi}{r}$, where p, q and r are positive integers, generates a tiling of the hyperbolic plane by tessellation when p, q and r satisfy the condition $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < \frac{1}{2}$.*

From this, we easily conclude that there are infinitely many tilings in the hyperbolic plane, each one generated by tessellation from a regular convex polygon P provided that the number p of sides of P and the number q of copies of P which can be put around a point A and exactly covering a neighbourhood of A without overlapping satisfy the relation: $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$. The numbers p and q characterize the tiling which is denoted $\{p, q\}$ and the condition says that the considered polygons live in the hyperbolic plane. Note that the three tilings of the Euclidean plane which can be defined up to similarities can be characterized by the relation obtained by replacing $<$ with $=$ in the above expression. We get, in this way, $\{4, 4\}$ for the square, $\{3, 6\}$ for the equilateral triangle and $\{6, 3\}$ for the regular hexagon.

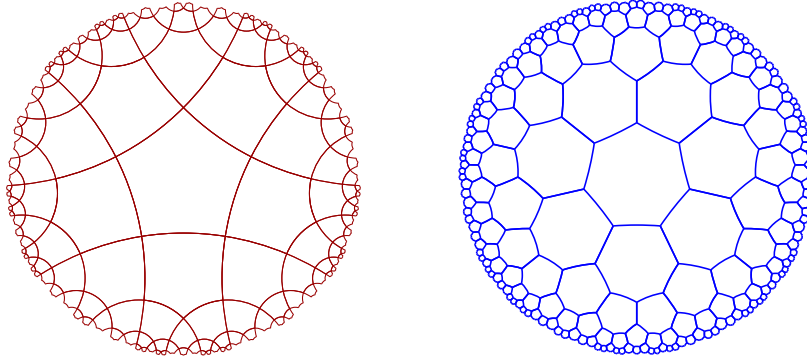


Figure 2 Left-hand side: the pentagrid. Right-hand side: the heptagrid.

In the paper, we shall focus our attention on the simplest tilings which can be defined in this way in the hyperbolic plane: $\{5, 4\}$ and $\{7, 3\}$. We call them the **pentagrid** and the **heptagrid** respectively, see Figure 2.

From Figure 2, it is not clear how to navigate in these tilings, even in the pentagrid or the heptagrid which are the simplest of the infinite family of tilings $\{p, q\}$. We introduce such a navigation tool in our next subsection. As a consequence of the existence of this tool, we can count the tiles of any tiling $\{p, q\}$.

4.2 Pentagrid and heptagrid

The left-hand side picture of Figure 3 indicates a recursive splitting of a quarter \mathcal{Q} of the hyperbolic plane which generates the pentagrid. The idea is that we place a vertex V of a rectangular regular pentagon P_0 at the corner of \mathcal{Q} , in such a way that the edges of P_0 which meet at V are along the sides of \mathcal{Q} . These edges are marked **1** and **5** on the figure.

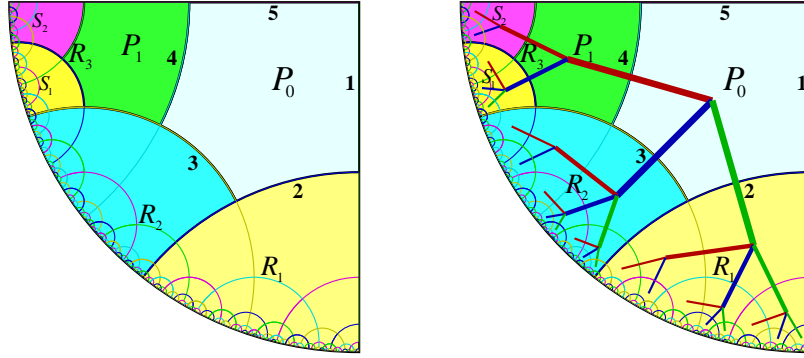


Figure 3 *The splitting of a quarter of the hyperbolic plane which generates the pentagrid. On the right-hand side: the tree structure which spans the tiling restricted to the quarter.*

Next, we consider the complement of P_0 in \mathcal{Q} . It can be split in the three regions labelled R_1 , R_2 and R_3 on the figure. The regions R_1 and R_2 are copies of a quarter: R_1 is obtained from \mathcal{Q} by the shift along side **1** moving V onto the corner of R_1 . Similarly, R_2 is obtained from \mathcal{Q} by the shift along **4** moving V on the corner of R_2 . What now remains is R_3 . Inside R_3 , P_1 is the reflection of P_0 in **4**. The complement of P_1 in R_3 is split into S_1 and S_2 . The region S_1 is again a quarter obtained from the quarter \mathcal{Q}_1 defined by **5** and **4** which contains P_1 by the shift along the side of P_1 which is opposite to **4**, the shift moving the corner of \mathcal{Q}_1 onto that of S_1 . Now, it is not difficult to see that S_2 is the image of R_3 by the shift along **5** which moves V onto the corner of R_3 . We say that the regions R_3 and S_1 are **strips**. Now we have two kinds of regions: quarters and strips. Next, Figure 3 allows us to split it into quarters and strips again, producing two or three tiles each time the process is applied to a quarter or a strip. The tree structure associated to this recursive process is underlined by the right-hand side picture of Figure 3. It is not difficult to see that the tree is generated by the following rules: $B \rightarrow BW$ and $W \rightarrow BWW$, considering that the tree has two kind of nodes, **white** and **black** ones. For each kind of node,

the rules indicate what is the kind of its sons and in which order.

Now, the tree allows us to number the tiles in a quarter: the root, which is level 0, receives number 1. Then, on level 1, the tiles receive numbers 2 to 4, running from left to right on the level. This is repeated on each level. It can be proved that on the level n , the leftmost tile receives the number f_{2n} and the tiles on this level are numbered from that number up to $f_{2n+2}-1$ which is given to the rightmost tile, where f_n is the Fibonacci sequence with initial conditions $f_0 = f_1 = 1$. This comes from the above rules and the details of the proof can be seen in [8] where an important property of this numbering allows us to construct efficient navigation tools. Again, we refer to [4, 8] for the exact proofs and a detailed account on the navigation. We call **Fibonacci tree** the tree obtained by the splitting process above described and illustrated by the right-hand side of Figure 3.

4.3 Extension of the splitting to other tilings

This *splitting method*, see [5, 8], can be extended to all tilings $\{p, q\}$. We indicate how it works for the heptagrid by Figure 4, the reader being referred to [5, 8] for explanations and more information.

Let us shortly indicate the basic patterns used in the case of the heptagrid. This time, we consider an angular sector S_0 defined by the intersection of two rays. These rays follow special lines adapted to the heptagrid which we call the **mid-point** lines, as they pass through mid-points of contiguous edges of the tiles. We consider that the sector contains the tiles which may have at most one vertex outside the rays with respect to the sector. In Figure 4, left-hand side, the 'big' copy of S_0 defined by the rays ℓ_1 and ℓ_2 and headed by τ contains two 'small' copies of S_0 : the first one is headed by τ_1 and defined by the rays ℓ_2 and m_1 ; the second one is headed by τ_2 and defined by the rays m_1 and m_2 . We have a second region, S_1 , which we again call a **strip**: it is headed by τ_3 and defined by the rays m_2 and ℓ_1 . Note that the lines which support these rays are non secant. It is not difficult to see that the line supporting the edge s_3 is their common perpendicular. Note that the copy of S_0 headed by τ_1 is obtained from the copy of S_0 headed by τ by the shift σ illustrated by the red arrow of the left-hand side of Figure 4.

Now, it is not difficult to see that S_0 is spanned by the same tree as the tree spanning \mathcal{Q} , see the right-hand side of Figure 4.

This identical spanning tree for the pentagrid and for the heptagrid is not a particular feature of both these tilings. It can be generalized to an infinite family of tilings, the tilings $\{p, 4\}$ and $\{p+2, 3\}$, with $p \geq 5$. The tilings $\{p, 4\}$ consists of the tessellations based on a regular rectangular polygon while the tilings $\{p+2, 3\}$ consists of those based on a regular polygon with angle $\frac{4d}{3}$, d being the measure of the right angle. We have that for each p with $p \geq 5$, the same tree spans the tilings $\{p, 4\}$ and $\{p+2, 3\}$. Of course, for $p > 5$ the tree is no more the Fibonacci tree: it is another tree, connected with another recurrent sequence, associated with an algebraic number, see [5, 8].

We would like to remark that, as proved in [8], the splitting method can also be applied to all tilings $\{p, q\}$ and provides a tree which spans the tiling. This tree is more complex than the ones we devised for the tilings $\{p, 4\}$ or $\{p+2, 3\}$.

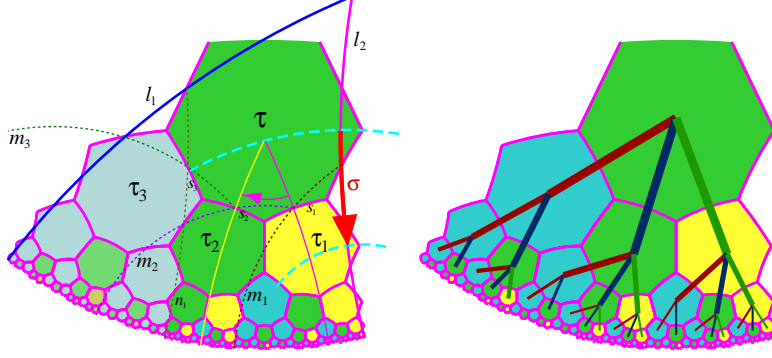


Figure 4 *The splitting of a sector of the hyperbolic plane which generates the heptagrid. Notice the underlying tree, the same one as for the pentagrid.*

Now, we have all elements in order to see how the infinite numeral system can bring in a more precise information to the picture which was described in this section. Before turning to this study, let us mention a few applications of the pentagrid and the heptagrid. Among those quoted in [9], let us mention the colour chooser using the heptagrid, see [1], the Japanese keyboard for cellphones using the pentagrid, see [12] and the communication protocol between tiles of the pentagrid or the heptagrid, see [7, 11].

5 Even and odd splittings and their applications

First, we look at the application of the infinite numeral system to the pentagrid and to the heptagrid. In Subsection 5.3, we shall see how to generalize these results to the family of tilings $\{p, 4\}$ and $\{p+2, 3\}$, and in Section 5.4, we look at the application of the infinite numeral system to the tilings $\{p, q\}$.

5.1 Pentagrid and heptagrid

In the case of the pentagrid, it is easy to see that we have two ways to realize the tiling. One way is based on the observation that the whole tiling can be split into exactly four quarters as illustrated on the left-hand side picture of Figure 5. Call this way the **even splitting**.

The other way consists in choosing a tile which will be called the **central tile** and then to notice that the complement of the tile in the plane can be split into exactly five quarters, see the left-hand side of Figure 6. Call this second way the **odd splitting**. Both ways are thoroughly explained in [8].

Now, we can see that each quarter has exactly the same number of tiles as

they are copies of each other which is based on a geometrical property. But what is this number of tiles? In [13], it was said that, 'from the enumeration property mentioned in Subsection 4.2, we can see that the number of tiles contained in each quarter is $\textcircled{1}$ '. We can consider this estimation as a *first approximation* of the expected number of tiles. This first approximation shows the same characteristics as the rule on infinity in traditional calculus: $\infty + \infty = \infty$. As remarked by Yaroslav Sergeyev in [20], this is analogous to the rule 'many' + 'many' = 'many' in Pirahã's numeral system consisting of 1, 2 and 'many', see the above paper and [3] for more information on Pirahã, a primitive tribe living in Amazonia.

Let us look closer at what happens and how the infinite numeral system can give us a more precise information.

In Subsection 4.2, we indicated that the number of nodes of the Fibonacci tree which stand at the level n is f_{2n+1} . From this, it is not difficult to prove that the number of nodes of the Fibonacci tree which are on a level m with $m \leq n$ is $f_{2n+2}-1$. Now, the Fibonacci tree is an infinite tree and, as it has a finite bounded branching, its height is also infinite. As we can assign a number to each tile of a quarter of the pentagrid, it is reasonable to assume that we have at most $\textcircled{1}$ tiles in a quarter. This leads us to consider the set \mathcal{F} of numbers n such that $f_n \leq \textcircled{1}$, extending the Fibonacci sequence to infinite indices by simply assuming that the induction definition $f_{n+2} = f_{n+1} + f_n$ still applies to infinite indices and to the consequently infinite terms of the extended sequence. Notice that explicit values can be obtained by using the expression:

$f_x = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^x - \left(\frac{1-\sqrt{5}}{2} \right)^x \right)$, where x can take any positive integer values, infinite ones being included. Define μ as the number of elements of \mathcal{F} . We know that we can consider μ as the greatest element of \mathcal{F} . This gives us that $f_\mu \leq \textcircled{1} < f_{\mu+1}$. Consequently, if $\vartheta = \frac{1+\sqrt{5}}{2}$, we have that $\mu \asymp \log_{\vartheta} \textcircled{1}$.

Now, let us remark that if we apply the rules defining the Fibonacci tree to a black node, we obtain another tree, which we call the **smaller** Fibonacci tree. By contrast, we call the Fibonacci tree issued from a white node the **standard** Fibonacci tree. It is not difficult, using the rules which define a Fibonacci tree, to prove that the number of nodes on the level n of a smaller Fibonacci is f_{2n} , so that the number of nodes of the level m with $m \leq n$ is f_{2n+1} .

Now, as the standard Fibonacci tree is clearly a faithful realization of the properties of the Fibonacci sequence, we can see that a smaller Fibonacci tree is indeed included in a standard one of the same height: the inclusion is even true at each level. We may assume that once the counting of a level x is performed, it goes to the end of level x : if the first nodes of level x have sons, there is no reason to consider that the last nodes of the same level have no son. These remarks have the following impact on μ . As a consequence, if μ were odd, a smaller Fibonacci tree of height μ could be realized by not a standard Fibonacci tree of the same height, a contradiction with the previous assumption. Accordingly, we may assume that μ is even, so that we can write $\mu = 2\eta$.

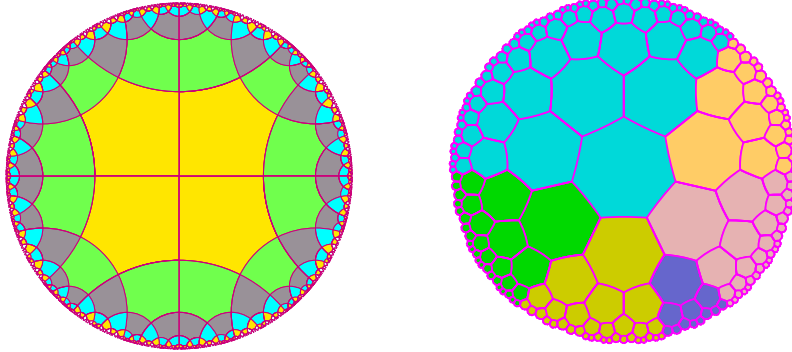


Figure 5 *The even splitting in the pentagrid, left-hand side, and in the heptagrid, right-hand side.*

Accordingly, we can consider that the number of tiles in a quarter of the pentagrid is $W = f_\mu - 1$, so that the eight of the spanning tree of the quarter is η . And so, in the even splitting of the pentagrid we get $4W$ tiles, while we get $1+5W$ tiles in the odd splitting. This is indicated in Theorem 2 and in Table 1.

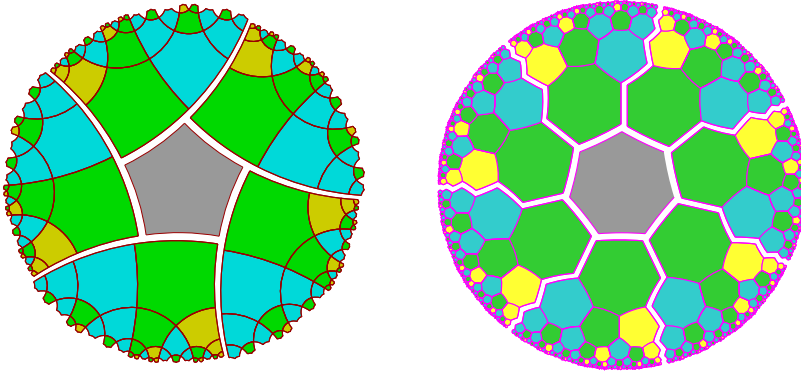


Figure 6 *The odd splitting in the pentagrid, left-hand side, and in the heptagrid, right-hand side.*

In the case of the heptagrid, the odd splitting can easily be established. It is illustrated by the right-hand side picture of Figure 6. This gives us $1+7W$ tiles.

The even splitting can also be defined for the heptagrid. We display three sectors whose heading tiles share a common vertex V , as illustrated by the right-hand side picture of Figure 5. Now, we have to take into account the heights of the Fibonacci trees involved in the splitting, which are not the same for each part. In order to make things comparable, we cannot take V as a centre, as we have no simple geometrical transformation which would allow us to compare the sectors. And so, instead of V , we take as the centre, one of the tiles which has V among its vertices, say T . We may consider that T is the light blue cell in the right-hand side picture of Figure 5, also the central cell of the figure. The sector headed by T contains $1+2W+B$ tiles, where B is the number of tiles

spanned by a smaller Fibonacci tree. We can easily see, that W is the number of tiles of the four other sectors headed by a tile which shares an edge with T . We remain with a sector whose height is $\eta-1$. Let W_1 be the number of tiles spanned by a standard Fibonacci tree of height $\eta-1$. Note that $W_1 = f_{2\eta-1} - 1$ and that $B = f_{2\eta}$, so that $W_1 + B = f_{2\eta+1} - 1 = W$. Accordingly, we find again $1+7W$ tiles.

Table 1 Table of the total area observable through the even and the odd splittings for the pentagrid and for the heptagrid. Remember that \mathbf{d} is the measure of the right angle.

	pentagrid	heptagrid
even splitting	$4W \cdot \mathbf{d}$	$\frac{14}{3}W \cdot \mathbf{d} + \frac{2}{3}\mathbf{d}$
odd splitting	$5W \cdot \mathbf{d} + \mathbf{d}$	$\frac{14}{3}W \cdot \mathbf{d} + \frac{2}{3}\mathbf{d}$

And so, we find a difference with the estimation of [13] which was based on a less precise estimation on the number of tiles. As a consequence, the number of tiles do not lead to the same observable area in the case of the pentagrid and in the case of the heptagrid.

Theorem 2 Let W be the number of tiles in a standard Fibonacci tree of height η where $f_{2\eta} \leq \mathbf{1} < f_{2\eta+1}$. The number of tiles of the pentagrid which can be observed with the help of the infinite numeral system is of the form $4W$ when using the even splitting while it is of the form $1+5W$ in the odd splitting. In the case of the heptagrid, the number of tiles is the same under the even or the odd splitting, and it is $1+7W$ in both cases. The total area covered by the tiles is given by the following table, see Table 1. Denote by P_e , P_o the total area which is observable in the pentagrid under the even, odd splitting respectively. Denote by H the total area which is observable in the heptagrid. Then we have that $P_e < H < P_o$ and, more precisely, $H - P_e = 2(P_o - H)$.

Now, we turn to the total area which can be observed under each splitting. Proof. Remember that the area of a triangle is the complement to the sum of the angles in order to obtain a straight angle. Remember that a straight angle is two right angles. We denote the right angle by \mathbf{d} . From this, we obtain that the area of a regular rectangle pentagon is $3.2\mathbf{d} - 5\mathbf{d}$ and so it is \mathbf{d} . Similarly, the area of a regular heptagon with angle $\frac{4\mathbf{d}}{3}$ is $5.2\mathbf{d} - 7 \cdot \frac{4\mathbf{d}}{3}$, so that it is $\frac{2\mathbf{d}}{3}$. From this we obtain the values indicated in Table 1. The values of P_e , P_o and H easily lead to the relation $H - P_e = 2(P_o - H)$, as the usual operations on numbers also hold for the infinite ones. This completes the proof of the theorem. ■

It is interesting to notice that the total area which is observable in the case of the heptagrid is in between the total area in the case of the pentagrid under the even or the odd splitting. It is also interesting to notice that the odd splitting

provides a significantly larger area in the case of the pentagrid, although the number of tiles is significantly less. This corresponds to the fact that the area of the regular rectangle pentagon is significantly bigger than that of the heptagon with the angle $\frac{4d}{3}$.

At last, it is worth noticing that the numbers of tiles $4W$ and $5W+1$ and $7W+1$ which we obtained are all bigger than \mathfrak{D} . This easily comes from the fact that $f_n < 3f_{n+1}$ which is generalized to infinite indices and from the definition of $W = U_{2\eta}$. From the very definition of $\mu = 2\eta$, we have that $U_\mu \leq \mathfrak{D} < U_{\mu+1}$ and, clearly, $U_{\mu+1} < 3U_\mu = 3W$. This property will turn out to be true in all the situations we consider in the paper.

Before turning to the generalization of the result of Theorem 2 to other tilings of the same family, we first remark that we obtain the same results if we change the place of the central tile in an odd splitting or of the central vertex in an even splitting. The second remark is an interesting generalization of the odd splitting which gives similar results¹.

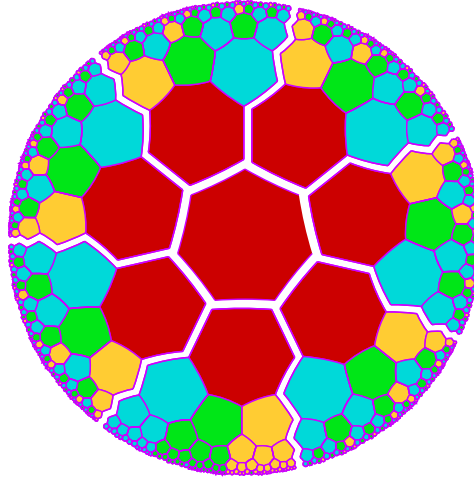


Figure 7 *Illustration, in the heptagrid, of the principle of the considered generalization: fixing a ball around the central tile, the tiles in red, the darkest tiles of the figure. Here, the radius of the ball is 1.*

Figure 7 shows the principle of a family of odd splittings which generalizes the one which we performed. The figure illustrates a particular case in the heptagrid. The general definition, which applies to both the pentagrid and the heptagrid is as follows.

First, we say that a tile A is at a **distance** k in tiles from a tile B , if there is a sequence T_0, \dots, T_h of tiles with $T_0 = A$, $T_h = B$, T_i and T_{i+1} share a side for $i \in \{0..h-1\}$ such that $k = h$ and that there is no such sequence for $h < k$.

¹It is my pleasure to thank Yaroslav Sergeyev for two questions he raised after reading the initial version of the paper. It is now the occasion to give the answers.

Then, from this we define a **ball of radius r around a tile T** , with r a positive number, as the set of tiles whose distance in tiles from T is at most r .

Now, we fix a ball of radius r around the central tile. And then, we split what remains in the space into sectors of the two kinds defined in Figures 3 and 4. The possibility to perform such a splitting is proved in [6, 8]. From these studies, we know that the number of tiles in the ball is $5f_{2r-1}+1$ in the pentagrid and $7f_{2r-1}+1$ in the heptagrid. Now, the number of sectors we need, is defined by the set of tiles which are outside the ball and also in contact with a tile of the ball. The number of these latter tiles is $5f_{2r+1}$ in the pentagrid and $7f_{2r+1}$. We have two kind of sectors which correspond to the one defined Figures 3 and 4, say white, black ones spanned by a standard, smaller Fibonacci tree respectively. As each node produces one black son exactly, the number of black nodes on the level $n+1$ of a standard, smaller Fibonacci tree is the number of nodes on the level n and so it is f_{2n+1} , f_{2n-2} respectively. This gives us f_{2n+2} , f_{2n-1} white nodes respectively.

We know the number of tiles spanned by each kind of sector. However, we do not start from a sector whose heading tile shares a side with the central cell but at a distance r from it. Accordingly, the height of such a sector is no more η , it is $\eta-r$. Accordingly, each white sector contains $W_r = f_{2(\eta-r)+1} - 1$ tiles and each black one contains $B_r = f_{2(\eta-r)}$ of them. Note that this is conformal to Postulate 3, as clearly a sector spanned by a standard Fibonacci tree strictly contains isometric images of itself. The definition of the height of such a sector allows us to measure precisely the difference with the central sector.

Now, call such an odd splitting an **r -augmented odd splitting**. This allows us to state the following result:

Theorem 3 *Under an r -augmented odd splitting, the number of tiles of the pentagrid which can be observed with the infinite numeral system is given by: $5f_{2r-1}B_r+5f_{2r}W_r+5(f_{2r}-1)+1$ for the pentagrid and for the heptagrid, it is given by: $7f_{2r-1}B_r+7f_{2r}W_r+7(f_{2r}-1)+1$. Accordingly, the observable total area is $(5f_{2r-1}B_r+5f_{2r}W_r)\mathbf{d}+(5(f_{2r}-1)+1)\mathbf{d}$ for the pentagrid and in the heptagrid it is given by: $\frac{14}{3}(f_{2r-1}B_r+f_{2r}W_r)\mathbf{d}+\frac{14}{3}(f_{2r}-1)\mathbf{d}+\frac{2}{3}\mathbf{d}$.*

Note that when $r = 0$, and setting $f_1 = 0$, we find again the results of Theorem 2.

Theorem 3 shows that, in this case too, the total area which is observable in the pentagrid is greater than that which is observable in the heptagrid although more tiles are observable in the latter than in the former.

5.2 General scheme for the splittings $\{p, q\}$

In the next sub-sections, we shall always follow the same scheme which can be formulated as follows.

First, we remind the splitting of the tiling. At this stage, we shall see that the general case can be split into several general situations: the tilings $\{p, 4\}$ and $\{p+2, 3\}$ which generalize most properties of the pentagrid and the heptagrid,

including the connection between this tilings; the tilings $\{p, q\}$ when q is even and then, when q is odd. We shall see that for this latter case, we have two solutions.

In this splitting, a region S_0 will play the most important role: we shall call it the **basic region**, and its spanning tree, the tree of the splitting will be denoted by \mathcal{T} . In all cases, the splitting induces a polynomial of the splitting from which we deduce a recurrent polynomial equation satisfied by the number u of nodes which are on the level n of \mathcal{T} . Of course, we shall extend the recurrent equation to infinite indices and, consequently, to the infinite values of u_n corresponding to infinite n 's. We shall then consider the number U_n of nodes on the levels m of \mathcal{T} with $m \leq n$. We know that $U_n = \sum_{i=0}^n u_i$. With the help of these numbers,

also extended to infinite integral indices, we can define the number η of positive integers x , finite or infinite, such that $U_x \leq \mathbb{1}$. This number will be considered as the height of \mathcal{T} . We shall immediately get that the number of tiles observable in S_0 is given by $W = U_\eta$. There will be no inconvenience to denote by the same letter W the different infinite numbers denoted by U_η , where U_x refers to the sequence u_n , u_n following different equations, depending on the splitting considered for the same tilings or for different ones. In each sub-section or sub-subsection, the meaning of S_0 , \mathcal{T} , u_n , U_n , η and W will be the same. Thanks to W and to the splitting of the tiling, it will be possible for us to give an expression for the number of tiles which are observable with the infinite numeral system based on $\mathbb{1}$.

Also, we shall compute the area α of the regular polygon on which the tiling is constructed, which we shall call the **basic polygon**. From this computation and from the area of a tile, we shall be able to compute a precise expression for the observable total area. This will allow us to compare this area for various splittings of the same tilings or for different tilings when this comparison will make sense.

5.3 Generalization to $\{p, 4\}$ and to $\{p+2, 3\}$

This sub-section is more different than what was presented in [13].

As announced in the Sub-section 5.2, this sub-section gives a direct generalization of what was done in the case of the pentagrid and of the heptagrid. The important common point with Subsection 5.1 is that the splittings of the tilings $\{p, 4\}$ and $\{p+2, 3\}$, with $p \geq 5$, are spanned by the same tree. It is no more the Fibonacci tree, but the new tree has several common properties with the Fibonacci tree.

We have again two kinds of nodes, corresponding to the two kinds of regions involved in the splitting: again we call them **black** and **white**, denoted by B and W respectively, the white nodes being attached to the bigger region. The tree which spans the big region is called **standard** and that which spans the small region is called **smaller** and the standard tree strictly contains the smaller tree. The rules are now $W \rightarrow BW^{p-3}$ and $W \rightarrow BW^{p-4}$, with $p \geq 5$.

This allows to easily infer the splitting of the tiling in these cases.

As proved in [8], the polynomial of the splitting is $X^2 - (p-2)X + 1$ and so, the number u_n of nodes on the level n of the tree satisfies the following recurrent equation: $u_{n+2} = (p-2)u_{n+1} - u_n$. The sequence is increasing with n and we have that $u_n \asymp \beta^n$, with $\beta = \frac{p-2 + \sqrt{p(p-4)}}{2}$. For a standard tree, we have $u_{-1} = 0$ and $u_0 = 1$, and for a smaller tree, we have $u_{-1} = u_0 = 1$. We extend the sequence to any integer, finite or infinite by the recurrent equation. Note that from the recurrent equation, using the standard representation involving a square 2×2 -matrix, and β being the biggest real root of $X^2 - (p-2)X + 1$, we can prove the following expression of u_n : $u_n = \frac{\beta}{\beta^2 - 1} \left(\beta^{n+1} - \frac{1}{\beta^{n+1}} \right)$, where n is a natural number, the result of the computation being also a positive integer. Extending n to positive infinite integers, this formula allows us to give an expression for u_n when it is infinite.

Together with W , we have to estimate the number of nodes of another tree \mathcal{A} : the one which spans the other region of the splitting which is called \mathcal{S}_1 , see [8]. This time, we denote by v_n the number of nodes in the level n of \mathcal{A} and by V_n , the number of nodes in the levels m with $m \leq n$ of \mathcal{A} . We know that v_n follows the same recurrent equation as u_n with different initial values: $v_0 = 1$ and $v_1 = u_1 - 1$. We also know that $V_m < V_n$, and we denote by B the number V_{φ_p} , observing that $B < W$ as, clearly, $\mathcal{A} \subset \mathcal{T}$, the inclusion being proper.

Now, we can see that for the tiling $\{p, 4\}$, we have a sharp difference between the even splitting which provides us with $4W$ tiles and the odd one which yields $1+pW$ tiles. Now, in the case of the tiling $\{p+2, 3\}$, this time we again shall find the same number of tiles for both the even and the odd splittings.

Indeed, the decomposition we observed for the even splitting of the heptagrid works word by word. For the same reason, we fix the central cell C as one of the three tiles heading a sector which share a common vertex. Then, around C , we have $p-1$ sectors spanned by the standard tree, one spanned by the standard tree of height φ_p-1 , which gives $W_1 = U_{\varphi_p-1}$ tiles. Now, the sector which is head by C also contains a region which is spanned by the smaller tree, for the same reason as in the heptagrid. This gives B tiles which, added to the W_1 tiles already mentioned give us W tiles as $W = B + W_1$, which is geometrically clear from Figure 5: indeed, the smaller region can be obtained from the smaller region by deleting a copy of the standard one. This can easily be seen on the spanning tree: the root of the standard tree has $p-2$ sons while the root of the smaller tree has $p-3$ of them, and the difference is a white tree. Accordingly, we obtain $1+(p+2)W$ tiles in the even splitting of the tiling $\{p+2, 3\}$. In the odd splitting of the same tiling, we again find $1+(p+2)W$ tiles for the same reason as in the heptagrid. And so, we can see that the situation which we observed in the heptagrid is generalized to all the tilings $\{p+2, 3\}$.

Now, we have to compute the observable total area in all cases.

In the general case of a regular polygon with p sides and of vertex angle $\frac{2\pi}{q}$, splitting the area of the polygon into $p-2$ triangles, we find that the area is

$(p-2)2\mathbf{d}$ minus the sum of the interior angles, *i.e.* $p \cdot \frac{2\pi}{q} = p \cdot \frac{4\mathbf{d}}{q}$. When $q = 4$, this gives us $(p-4)\mathbf{d}$. When $q = 3$, with a regular polygon with $p+2$ sides, this gives us $(p-4)\frac{2\mathbf{d}}{3}$. This shows us that again, the area of the regular rectangular polygon with p sides is bigger than the regular polygon with the angle $\frac{4\mathbf{d}}{3}$ with $p+2$ sides. Despite the bigger number of tiles in the tiling $\{p+2, 3\}$, the total area is still bigger for the tiling $\{p, 4\}$ under the odd splitting, as we can see from Table 2. As in the case of the heptagrid and the pentagrid, the total area for the tiling $\{p+2, 3\}$ is the same for both splittings and it takes an intermediate value between the total area for the tiling $\{p, 4\}$ under the even splitting and its total area under the odd splitting. This time, the total area of the tiling $\{p+2, 3\}$ is much closer to the total area of the tiling $\{p, 4\}$ under the odd splitting.

Theorem 4 *In the tiling $\{p, 4\}$, the number of tiles which can be observed with the infinite numeral system is given by $4W$ under the even splitting and by $1+pW$ under the odd splitting. For the tiling $\{p+2, 3\}$, the corresponding number of tiles is $1+(p+2)W$ under both the even and the odd splittings. The total area which can be observed in each case is given by Table 2. Denote by P_e , P_o the total area which is observable in the tiling $\{p, 4\}$ under the even, odd splitting respectively and denote by H the observable total area in the tiling $\{p+2, 3\}$. Then, we have that $H - P_e = 2(P_o - H)$.*

Indeed, the computation which we can perform from Table 2 gives us that $H - P_e = \frac{2}{3}(p-4)^2W\mathbf{d} + \frac{2}{3}(p-4)\mathbf{d}$ and $P_o - H = \frac{1}{3}(p-4)^2W\mathbf{d} + \frac{1}{3}(p-4)\mathbf{d}$.

This completes the proof of Theorem 4. ■

Table 2 *Table of the total area observable through the even and the odd splittings for the tilings $\{p, 4\}$ and $\{p+2, 3\}$ tilings, \mathbf{d} being the measure of the right angle.*

	$\{p, 4\}$	$\{p+2, 3\}$
even splitting	$4(p-4)W.\mathbf{d}$	$(p+2)(p-4)W\frac{2\mathbf{d}}{3} + (p-4)\frac{2\mathbf{d}}{3}$
odd splitting	$p(p-4)W.\mathbf{d} + (p-4)\mathbf{d}$	$(p+2)(p-4)W\frac{2\mathbf{d}}{3} + (p-4)\frac{2\mathbf{d}}{3}$

At last, and not the least, it is also possible in this case to define an r -augmented odd splitting, both in the tilings $\{p, 4\}$ and $\{p+2, 3\}$, considering a ball of radius r around the central cell. However, the estimation of the number of observable tiles and their areas involve developments of [8] which we have no room to reproduce here.

5.4 The general case $\{p, q\}$

This case is more complex than the case of the tilings studied in Subsection 5.3 by the fact that the splittings in the case when q is even and in the case when it is odd are very different. Moreover, there is no simple way to define r -augmented odd splittings. And so, we shall look at the even and odd splittings only.

First, in Sub subsection 5.4.1, we remind the splitting of [8] in the case when q is even. Then, in Sub subsections 5.4.2 and 5.4.3, we deal with the case when q is odd. As announced in Sub-section 5.2, we shall offer two different splittings when q is odd, each one having its own merit.

5.4.1 The case when q is even

In that case, we define a **sector** \mathcal{S}_0 as the angular sector defined by taking a vertex V of the polygon P on which the tiling is constructed and the rays issued from V which supports the two edges of P which meet at V . We call V the vertex of \mathcal{S}_0 and P is its **head**. It is easy to see that the whole tiling is the union of q copies of \mathcal{S}_0 which share the same vertex.

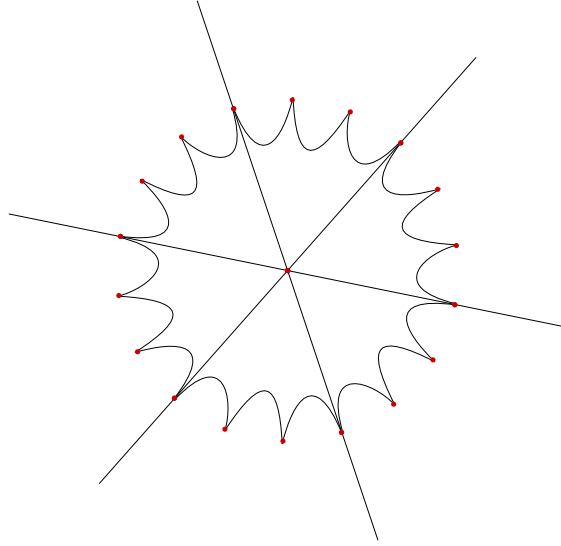


Figure 8 *The even splitting of the tiling $\{p, q\}$ when q is even.*

In [8] we indicate how the tiling $\{p, q\}$ can be split in order to prove that the tiling is combinatorial. From this we know that we can define a tree which spans the restriction of the tiling to a copy of \mathcal{S}_0 in the conditions given in the above definition of such a sector. We can see a representation of this splitting, which generalizes the notion of even splitting to the case of the tiling $\{p, q\}$ when q is even, in Figure 8.

In [8] we also indicate the polynomial of the splitting, which is given by $X^2 - ((p-3)(h-1)+1)X - h + 3$, with h defined by $q = 2h$, from which we obtain that the number u_n of nodes on the level n of the spanning tree satisfies the following recurrence equation: $u_{n+2} = ((p-3)(h-1)+1)u_{n+1} - (h+3)u_n$, with $u_{-1} = 0$ and $u_1 = 1$. From this, we define W , the number of tiles observable in \mathcal{S}_0 as indicated in Sub-section 5.2.

Now, for the evaluation of the number of tiles, note that the even splitting gives us that exactly q copies of \mathcal{S}_0 cover the hyperbolic plane with no overlapping. As the number of nodes in the spanning tree is W , we have that the number of tiles is qW . On another hand, the area of a tile is here given by $(p-2).2\mathbf{d} - p\frac{4\mathbf{d}}{q}$, so that the total area which can be observed in these condition is $2(pq-2(p+q))W\mathbf{d}$.

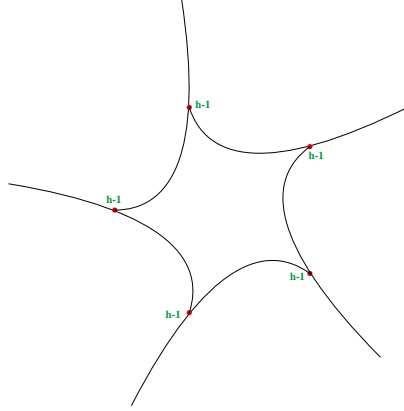


Figure 9 The odd splitting of the tiling $\{p, q\}$ when q is even. Note that $h-1$ indicates the number of copies of \mathcal{S}_0 which can be displayed in a fan rooted at the nearby vertex, between the two rays defined by the sides meeting at the vertex, as indicated in the figure.

Figure 9 indicates the splitting of the hyperbolic plane organized around a central tile. We can see that the complement in the hyperbolic plane of the central tile can be split into p regions which are copies of each other. Such a region is delimited by a vertex and two rays, each one supporting an edge of the central tile abutting the vertex. The angle between the two rays is $(h-1)\frac{4\mathbf{d}}{q}$ and so, we can see that there is there room for exactly $h-1$ copies of \mathcal{S}_0 , all of them with the same vertex. We say that such a region is a **fan** of $h-1$ copies of \mathcal{S}_0 . Accordingly, this time the number of tiles is $p(h-1)W + 1$. We again find the result we have found for the tilings $\{p, 4\}$. Also note that the total area which falls under observation is $(p(h-1)W + 1) ((p-2).2\mathbf{d} - p\frac{4\mathbf{d}}{q}) =$

$$2(h-1)\frac{pq-2(p+q)}{q}W\mathbf{d} + 2\frac{pq-2(p+q)}{q}\mathbf{d} = (h-1)\frac{s}{q}W\mathbf{d} + \frac{s}{q}\mathbf{d}, \text{ where we have put } s = 2(pq-2(p+q)).$$

5.4.2 The case when q is odd: a first solution

In this case, we can no more use the mid-point lines which we considered in the case when $q = 3$. There is a solution which is thoroughly described in [8], but which is different, in its principle from what was done in the case when $q = 3$ and also from the case when q is even. There is a new solution in [10] which we shall briefly describe in the next paragraphs.

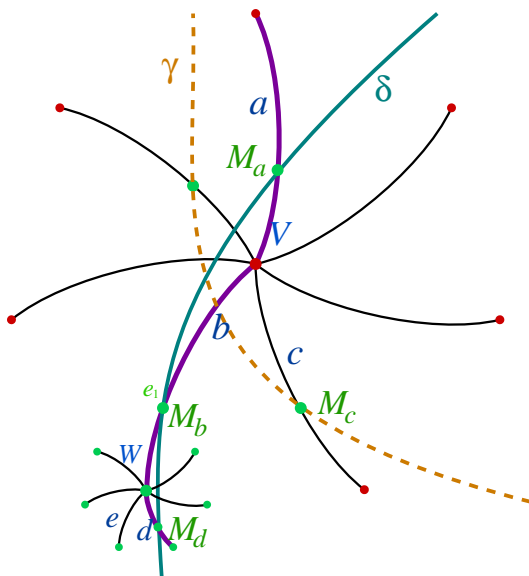


Figure 10 *The definition of the h -mid-point line.*

This solution consists in taking into consideration another mid-point line, what is called a **h -mid-point line** in [10], where $h = \lfloor \frac{q}{2} \rfloor$. This new line comes from the following consideration. Around a vertex V , we exactly have q copies of the polygon used for defining the prototile. Fix one edge a among the q ones abutting V . There are exactly two edges b and c abutting V which makes an angle $h \frac{2\pi}{q}$ with a . Let M_a , M_b and M_c be the mid-points of a , b and c respectively, see Figure 10. Consider W the other end point of b . We can find two edges d and e abutting W and making with b the same angle $h \frac{2\pi}{q}$. Let d be the edge which is on the other side of the line supporting b with respect to a : a is one half-plane defined by this line while b is in the other one. Let M_d be the mid-point of d . It is easy to see that the triangles M_aVM_b and M_bWM_d are

equal and, consequently, that the angles $(b, M_b M_a)$ and $(b, M_b M_d)$ are equal. As these angles are not on the same side of the line supporting b , the points M_a , M_b and M_d lie on the same line which is called an **h -mid-point line**. Clearly, M_a and M_c also define another h -mid-point line which is symmetric to this one under the reflection in the line δ supporting a .

This allows us to define what we shall call sector \mathcal{S}_0 when q is odd. Figure 11 illustrates the display of q copies of \mathcal{S}_0 around a central vertex. The figure also illustrates the construction we have just defined in the previous paragraph.

We can see that here too, the hyperbolic plane can be split into q sectors exactly.

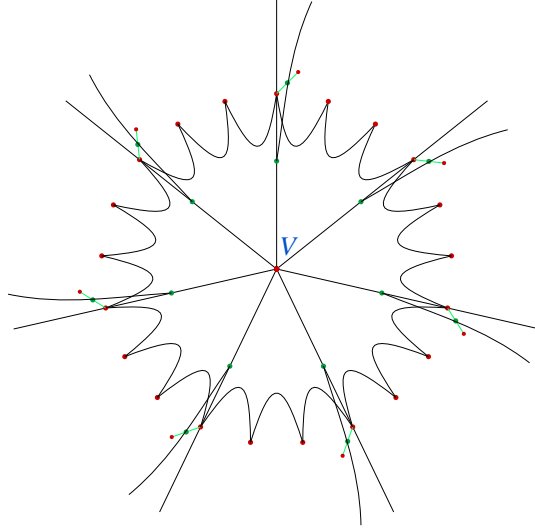


Figure 11 *The even splitting of the tiling $\{p, q\}$ when q is odd. Note the h -mid-point lines which define the q copies of \mathcal{S}_0 which share a common vertex. It goes from the mid-point of an edge abutting the central vertex and through the mid-point of an edge making an angle $h \frac{2\pi}{q}$ with the edge of the first mid-point we have just considered.*

As shown in [10], it can be proved that we need three types of region in order to obtain a combinatoric tiling, and we get that the polynomial of the splitting is $X^3 - ((p-3)(h-1)+1)X^2 - ((p-2)(h-1)-2)X - h+3$. This give rise to the following recurrence equation, which involves one more term in the right-hand side: $u_{n+3} = ((p-3)(h-1)+1)u_{n+2} + ((p-2)(h-1)-2)u_{n+1} - (h-3)u_n$, where u_n is the number of nodes which are on the level n of the spanning tree. This allows us to define W , the number of observable tiles in the sector \mathcal{S}_0 .

Consequently, we have qW tiles which can be observed in this way. Now, the computation of the area of the basic polygon is the same as in the case when q is even so that we find the same expression for the total area under observation, namely $sW\mathbf{d}$, with again $s = 2(pq-2(p+q))$.

Let us now consider the odd splitting for the tiling $\{p, q\}$ when q is odd.

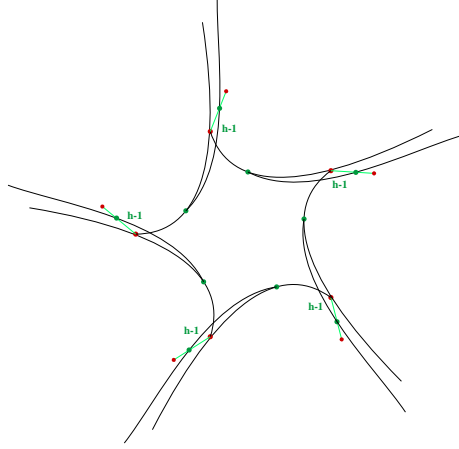


Figure 12 The odd splitting of the tiling $\{p, q\}$ when q is odd. Note the h -mid-point lines which define the q fans of $h-1$ copies of \mathcal{S}_0 which are displayed around a central tile.

Figure 12 illustrates the display of p fans of $h-1$ copies of \mathcal{S}_0 around a central tile. We note that this splitting is different from the one we considered when $q = 3$. The definition of the sectors under the general definition would define a region which is the union of two sectors in terms of those of the heptagrid. We shall soon go back to this point.

Accordingly, we can see that now we have $p(h-1)W + 1$ tiles under observation. Accordingly, the total area which is observable is defined by the same formula for the odd splitting as in the case when q is even.

We can now formulate the results:

Theorem 5 The number of tiles of the tiling $\{p, q\}$, when $q \geq 4$ is qW in the even splitting, independently of the parity of q , where $W = U_\mu$, and the total observable area is $sW\mathbf{d}$, where $s = 2(pq - 2(p+q))$. With the odd splitting, the number of tiles which can be observed is $p(h-1)W + 1$, where h is defined by $h = \lfloor \frac{q}{2} \rfloor$, corresponding to the possible observation of a total area which is $p(h-1)\frac{s}{q}W\mathbf{d} + \frac{s}{q}\mathbf{d}$.

As we can see from the areas indicated in the theorem, the odd splitting allows us to observe a much bigger area: $(h-1)\frac{s}{q}$ is a bit smaller than $\frac{1}{2}$, but closer to this value as q increases. Besides, p is at least 3, so that $(h-1)\frac{p}{q} > 1$, from which we conclude that $p(h-1)\frac{s}{q}W\mathbf{d} + \frac{s}{q}\mathbf{d} > sW\mathbf{d}$.

5.4.3 The case when q is odd: another solution

In [10], we indicate another splitting of the tiling $\{p, q\}$ in the case when q is odd. This gives an alternative proof that the tiling is combinatoric. The advantage is that the number of basic regions is now two instead of three in the previous solution, so that the polynomial of the splitting has degree 2. In fact, the new splitting comes from the fact that one region of the previous solution can easily be split into the two others. And this region is the generalization to the general case $\{p, q\}$, with q odd, of what was done in the heptagrid. If we go back to Figure 10, the dashed line γ is a rotated image of the line δ , and a sector is defined in this way in the previous solution. In the solution indicated in this subsection and which is detailed in [10], the sector is defined by the lines $M_a M_b$ and $M_a M_c$. This means that the second line is not a rotated image of the first one around the vertex V but the reflection of the first line in the bisector of the angle at V between the two sides of the polygon meeting at V . Note that this is the generalization of the definition of a sector given for the heptagrid. The second region is \mathcal{S}_1 as in the previous solution.

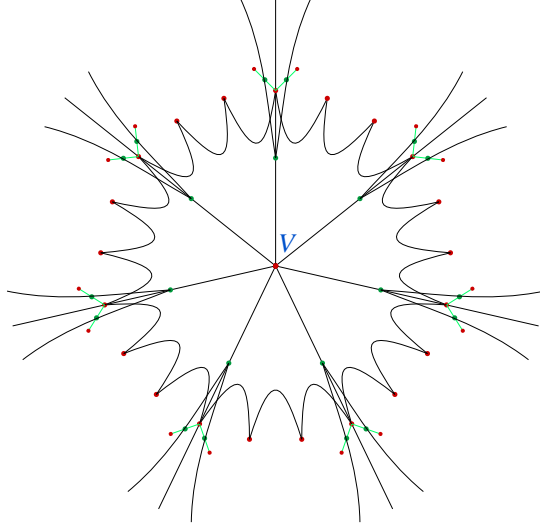


Figure 13 *The even splitting of the tiling $\{p, q\}$ when q is odd, alternative version. Note the h-mid-point lines which define the q copies of \mathcal{S}'_0 which share a common vertex. The union of the q copies of \mathcal{S}'_0 around V do not cover the plane.*

Now, following the computations of [10], the polynomial of the splitting is: $X^2 - ((p-3)(q-3)+1)X - q+7$, which is also very different from what we obtain in [8]. From the polynomial, we get the recurrent equation, again with two terms on the right-hand side: $u_{n+2} = ((p-3)(q-3)+1)u_{n+1} + (q-7)u_n$. Again, we define W as indicate in Sub-section 5.2, the basic region being now \mathcal{S}'_0 .

This time, we can see that there is a difference between the even and the odd splittings. Indeed, if we consider q copies of \mathcal{S}'_0 around V , they do not cover the hyperbolic plane. As we can see in Figure 13, in between two contiguous copies of \mathcal{S}'_0 , there is room for another copy of \mathcal{S}'_0 which is a proper part of \mathcal{S}'_0 . It is easy to see that the height of the tree spanning \mathcal{S}'_0 being η , the height of the tree spanning this other copy of \mathcal{S}'_0 is $\eta-1$. Let us set $W_1 = U_{\eta-1}$ and let us denote by \mathcal{S}_0^1 the set of tiles spanned the tree of height $\eta-1$. Then, in the even splitting, the plane is split into q copies of \mathcal{S}'_0 and q copies of \mathcal{S}_0^1 which cover exactly the plane with no overlapping, the number of tiles being $q(W+W_1)$. For the odd splitting, we have the same p copies of a fan of $h-1$ copies of \mathcal{S}_0 and $h-1$ of \mathcal{S}_0^1 , which means $2(h-1)$ copies of \mathcal{S}'_0 as \mathcal{S}_0 can exactly be split into two copies of \mathcal{S}'_0 . Accordingly we have $p(h-1)(W+W_1)+1$ tiles. Now, the area of the regular polygon is $2(pq-2(p+q))\frac{\mathbf{d}}{q}$ in both cases. This allows us to state the following result:

Theorem 6 *The number of tiles of the tiling $\{p, q\}$ in the alternative splitting, when $q \geq 5$ and q is odd, is $q(W+W_1)$ under the even splitting and $p(h-1)(W+W_1) + 1$ under the odd splitting, where $W_1 = U_{\eta-1}$ while $W = U_\eta$. The observable total area is given by $s(W+W_1)\mathbf{d}$ in the even splitting and by $p(h-1)\frac{s}{q}(W+W_1)\mathbf{d} + \frac{s}{q}\mathbf{d}$ in the odd splitting.*

As in the previous solution, we can notice that the odd splitting allows us to observe a bigger area than in the even one.

6 Conclusion

It is interesting to see the difference between the even and odd splitting in the study of the number of tiles which can be observed using the infinite numeral system. In all cases, we can count more than $\mathbf{1}$ tiles. Also, in all cases, the odd splitting gives more tiles and, accordingly a greater area. This is particularly striking when $q \geq 5$ in the tilings $\{p, q\}$.

While looking at the families $\{p, 4\}$ and $\{p+2, 3\}$ which have the same spanning tree, it is worth noticing that, despite the fact that in the odd splitting, the tiling $\{p+2, 3\}$ gives access to more tiles than the tiling $\{p, 4\}$, the total observable area is bigger with the tiling $\{p, 4\}$: this corresponds to the fact that the basic polygon is significantly bigger in this tiling. Another remarkable point is the fact that the ratio between the different areas we can observe is the same, independently of p .

Now, when q is odd, the observable total area is bigger in the second solution than in the previous one: it is twice the previous one for the even splitting and almost twice too for the odd splitting. This indicates the interest of the second solution which is also useful for its possibility to simulate action at a distance: indeed, in this splitting, each node of the tree has sons which correspond to a tile which is not in contact with the tile of the father, even by a vertex only.

It appears that the infinite numeral system based on ① gives a precise tool to measure properties which allow us to introduce a distinction in various splittings of the same tiling which, classically, are all equivalent. It would be interesting to explore the possibilities given by this system for other criteria.

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